Прикладные задачи нелинейной теории колебаний и волн

Изв. вузов «ПНД», т. 13, № 4, 2005

APPROXIMATE SYNCHRONIZATION OF CHAOTIC ATTRACTORS

Oscar de Feo, Martin Hasler

This work presents a dynamical phenomenon strongly related with the problems of synchronization and control of chaotic dynamical systems. Considering externally driven homoclinic chaotic systems, it is shown experimentally and theoretically that they tend to synchronize with signals strongly correlated with the saddle cycles of their skeleton; furthermore, when they are perturbed with a generic signal, uncorrelated with their skeleton, their chaotic behavior is reinforced. This peculiar behavior of approximate synchronization has also been called *qualitative resonance*, underlining the fact that such chaotic systems tend to resonate/synchronize with those signals which are qualitatively similar to an observable of their skeleton.

Introduction

Since the pioneering works of Afraimovich, Verichev, and Rabinovich [1], a considerable investigative effort has been dedicated to the problem of synchronization of chaotic dynamical systems as well as to the problem of their control. In this work, a dynamical phenomenon strongly related to these two problems is introduced and its analysis, both experimental and theoretical, is presented. In particular, it is shown that different dynamical models (ordinary differential equations) admitting chaotic behavior organized by a homoclinic bifurcation (here called Shil'nikov-like chaotic systems) tend to have a quite particular selective property when externally perturbed. Namely, these systems settle on a very narrow chaotic behavior, which is strongly correlated to the forcing signal, when they are slightly perturbed with an external signal which is similar to their corresponding generating cycle (GC). Here, the «generating cycle» is understood to be the saddle cycle colliding with the equilibrium at the homoclinic bifurcation or, in other words, the lowest period cycle embedded in the Shil'nikov-like chaotic attractor. On the other hand, when they are slightly perturbed with a generic signal, with no particular correlation with their GC, their chaotic behavior is reinforced. This peculiar sympathetic behavior of approximate synchronization has also been called «qualitative resonance» (QR) [2, 3]. This name wants to highlight the fact that such chaotic systems tend to *resonate* with those signals which are qualitatively similar to an observable of their corresponding GC. Indeed, when approximately synchronizing, the behavior of the forced system shrinks from a wide chaotic attractor to a very narrow one, which is very similar to a resonating torus which is shrinking from many modes to few ones. Moreover, this name has also been chosen to distinguish this phenomenon from proper synchronization; because, even if related to it, the observed behavior is not, rigourously speaking, strict synchronization.

The paper is organized as follows: first, the QR phenomenon is introduced and previous results [2, 3] about it are briefly recalled; second, a linear explanation of the phenomenon is given together with a geometrical conjecture of it; finally, the geometrical conjecture is confirmed by means of bifurcation analysis.

1. The Qualitative Resonance Phenomenon

Given a generic *error driven* dynamical system (Eq. (1)) admitting Shil'nikov-like chaotic behavior when there is no external control ($K = 0$):

$$\begin{aligned} \dot{x} &= F(x) + K\big(y - u(t)\big), \quad x \in \mathbb{R}^n,\, F : \mathbb{R}^n \mapsto \mathbb{R}^n \\ y &= H(x), \quad y \in \mathbb{R}^m,\, H : \mathbb{R}^n \mapsto \mathbb{R}^m,\, m < n \end{aligned} \tag{1}$$

the system is *slightly* (*i.e.* small value of $\|K\|$) perturbed with different kinds of perturbing signals $u(t)$, some strongly related to the GC of the Shil'nikov-like strange attractor and some not. For each given driving signal, the system's steady state is classified as follows: whenever the steady state is shrinking to a periodic solution or to a chaotic solution with a very small variance with respect to the GC, *i.e.* something very close to a limit cycle, the ensemble is said to *qualitatively resonate* or to *approximately synchronize* [2, 3]; on the other hand, whenever the steady state is chaotically wandering, the ensemble is said to not qualitatively resonate or to *anti-resonate/synchronize*.

More precisely, K is considered slightly perturbing the system if:

$$\max_{\substack{x \in \mathrm{SA} \\ u(t) = y(t),\, y \in \mathrm{GC}}} \Big\| K\big(y - u(t)\big) \Big\| < \frac{1}{\beta} E\Big[\big\|F(x)\big\| \,\Big|\, x \in \mathrm{SA} \Big], \quad \beta > 1$$

where $x, y \in \mathrm{SA}$ means that the state x or the corresponding output y are on the uncontrolled strange attractor (SA); $u(t) = y(t), y \in \mathrm{GC}$ means that the perturbing signal $u(t)$ is the time series of the output y while evolving on the GC; and, $E[\cdot]$ is the averaging operation. In simple words, the maximal external perturbation on the evolution of the system (LHS) must be at least β times smaller than the average natural evolution (RHS). Furthermore, the system is classified as *qualitatively resonating* whenever the following condition is satisfied:

$$\int_{t-nT_{\mathrm{G}}}^{t} \big(y(\tau) - u(\tau)\big)^2 d\tau < \frac{1}{\gamma} E\Big[\int_{t-nT_{\mathrm{G}}}^{t} \Big(y(\tau)\big|_{y \in \mathrm{SA}} - y(\tau)\big|_{y \in \mathrm{GC}} \Big)^2 d\tau \Big], \quad \gamma > 1,\, n \in \mathbb{N} \tag{2}$$

where T_{G} is the period of the GC. In simple words, a system is said to be resonant when the driven trajectories (LHS) are γ time closer to the perturbing signal $u(t)$ than what is expected to be the free strange attractor to the GC (RHS).

1.1. Experimental Evidence. In [2] experimental evidence of QR occurring in Shil'nikov-like chaotic systems has been reported. To have a feeling about the phenomena, here some of the results reported in [2] are shown. Fig. 1 reports the occurrence of QR in the Colpitts oscillator [4] perturbed with fine or coarse piecewise linear approximations of its observable $y = x_2$ while evolving on the GCs of different strange attractors, where the GCs have been obtained by numerical continuation [5, 6].

2. Working Principles of Qualitative Resonance

It is intuitive to explain the QR phenomenon in the case of resonance with the clean signal coming from the GC. Such a case correspond to a simple reconstruction of a periodic linear system [7], *i.e.* just a particular case of synchronization between chaotic systems. In fact, consider an autonomous nonlinear system like (1) with $K = 0$, and suppose that the system admits a periodic solution of period T

$$\widehat{x}(t) : \widehat{x}(t+T) = \widehat{x}(t), \, \dot{\widehat{x}}(t) = F(\widehat{x}(t)), \quad \widehat{y}(t) = H(\widehat{x}(t)),$$

then $\widehat{x}(t)$ is also a periodic solution of the nonautonomous system with $K \neq 0$, $\forall K : \, \dim(K) = n \times m$. Furthermore, under suitable conditions, $\widehat{x}(t)$ is a stable solution of the nonautonomous ($K \neq 0$) system independently from its stability in the autonomous ($K = 0$) system. Indeed, system (1) can be linearized around the periodic solution $\widehat{x}(t)$ leading to a periodic linear system

$$\begin{aligned} \dot{\delta x} &= A(t)\delta x - K \underbrace{(y - \widehat{y})}_{\delta y}, \\ \delta y &= C(t)\delta x. \end{aligned} \tag{3}$$

In control theory [8] it is known that if the couple $(A(t), C(t))$ is observable then the characteristic multipliers of system (3) can be arbitrarily assigned and, if the observation matrix $C(t)$ is a constant matrix, this can be done with a constant matrix K. Then, the stability of the solution $\delta x(t) = 0$ of system (3) corresponds to the asymptotic stability of the periodic solution $\widehat{x}(t)$ of system (1) provided that the Jacobian matrix $A(t)$ and the reference signal $\widehat{y}(t)$ are almost in phase. Namely, provided that the state $x(t)$ of system (1) univocally identifies the periodic time $t \mod T$ in system (3), since one is an autonomous nonlinear system while the other is a time-varying linear system. In a Shil'nikov-like strange attractor, this condition is implicitly satisfied since the phase changes randomly whenever the trajectory passes nearby the equilibrium bearing the homoclinic. Thus, if the gain matrix K is such that Shil'nikov's conditions are not violated, sooner or later the driving signal and the state of the driven system will be in phase, and the linear control theory will warrant the convergence to the GC. Concluding, the QR with the clean signal coming from the GC of a Shil'nikov-like strange attractor is a straight consequence of the random phase seeking of Shil'nikov-like chaos and of the periodic linear control theory.

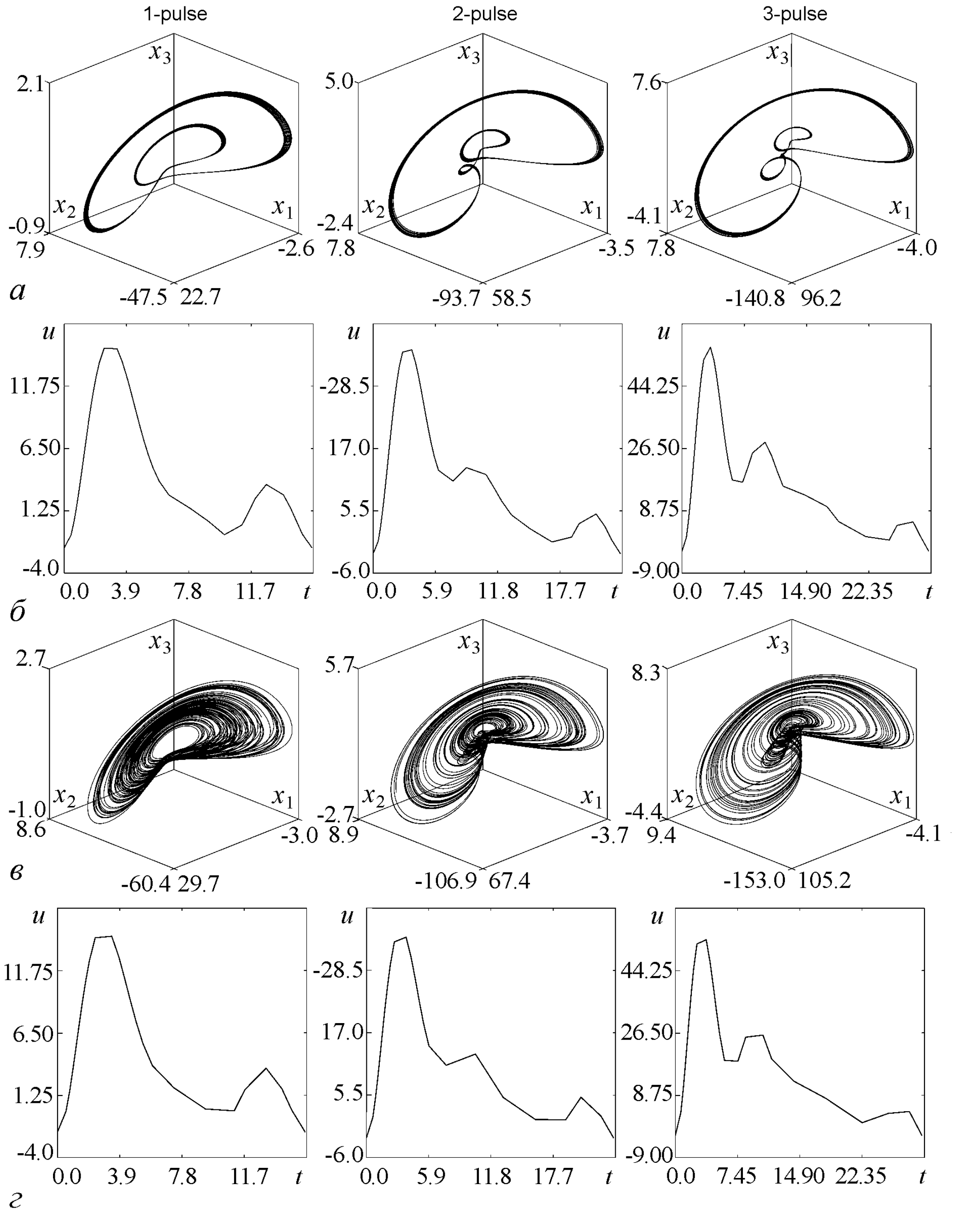


Fig. 1. Qualitative resonance for the Colpitts oscillator when driven with piecewise linear approximations of the observable $y = x_2$ corresponding to the n-pulse generating cycle of its strange attractors:*a* – qualitative resonance with a fine piecewise linear approximation; *b* – driving signals corresponding to *a*; *c* – qualitative anti-resonance with a coarse piecewise linear approximation; *d* – driving signals corresponding to *c*.

2.1. Geometric Conjecture. The above control theory argument is also useful to explain the occurrence of the QR phenomena for not so clean driving signals. When a feedback matrix gain K is chosen, the asymptotic *noise reduction ratio* (NRR) of the input noise for the filter given by Eqs. (3) is uniquely defined (periodic Kalman Filter). Therefore, feeding the system (1) with a signal $\widehat{y}(t) + \varepsilon_{in}$, when the signal and the system will be phase, will lead the system to shrink close to the periodic solution $\widehat{x}(t)$ as much as predicted by the NRR. Therefore, since a very simple generalization of a cycle is a Feigenbaum-like (period doubling) strange attractor, it follows that feeding the system (1) with a Feigenbaum-like signal coming from a strange attractor lying on a Möbius strip large ε will lead the system to shrink on a strange attractor NRR time narrower than that of the source, similarly as what shown in Figs. 1, *a*.

Taking into account the geometrical structure of a Shil'nikov-like strange attractor, *i.e.* a Matrioshka containing infinite self-similar Feigenbaum-like strange saddles (Fig. 2, *a*), the geometrical working principle of the QR is easily understandable. Because of the existence of infinite self similar skeleton saddle cycles that lie on the strange attractor manifold, also a cycle built by a piecewise composition of arcs of skeleton cycles which, consequently, lies on the manifold and satisfies the tachometric law on it, can be stabilized by a similar procedure as the one described above. Obviously, such a new cycle cannot be to much different from the skeleton cycles, because of its construction constraints. Thus, it will be «just a generalization» of the stereotype cycles of the skeleton leading to the QR phenomenon. A geometrical sketch of this working principle is given in Fig. 2, *b*.

Finally, due to the fact that a Shil'nikov-like strange attractors lie on an almost one-dimensional manifold which has a transversal attracting direction and is repulsive in the directions that are parallel to this manifold, and since it has been assumed that the feedback gain matrix K does not alter excessively the dynamic of the system, it follows that the minimal K which stabilizes the periodic solution $\widehat{x}(t)$ must mainly stabilize the repelling direction while leaving almost unaltered the dynamics in the already stable direction.

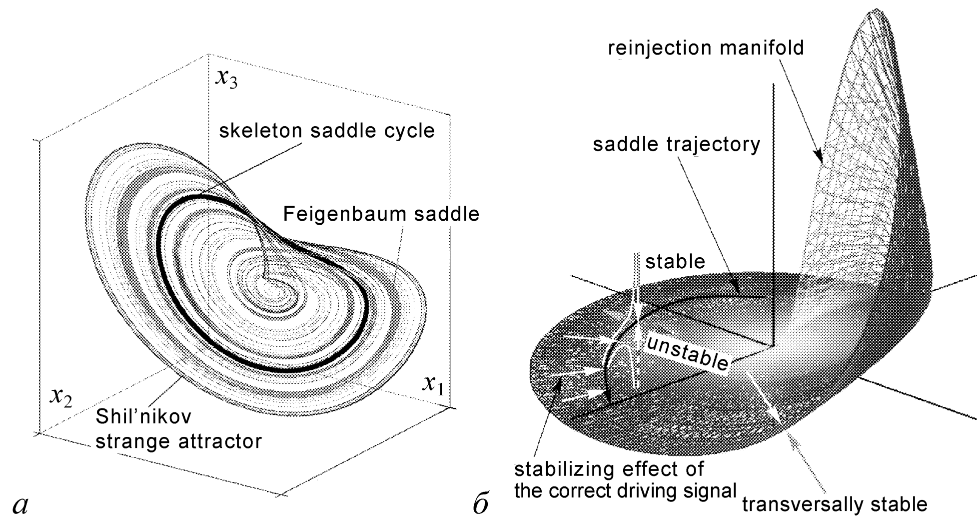


Fig. 2. Geometrical explanation of QR: *a* – Matrioshka structure of a Shil'nikov-like attractor; *b* – the driving forces of qualitative resonance in a Shil'nikov-like strange attractor.

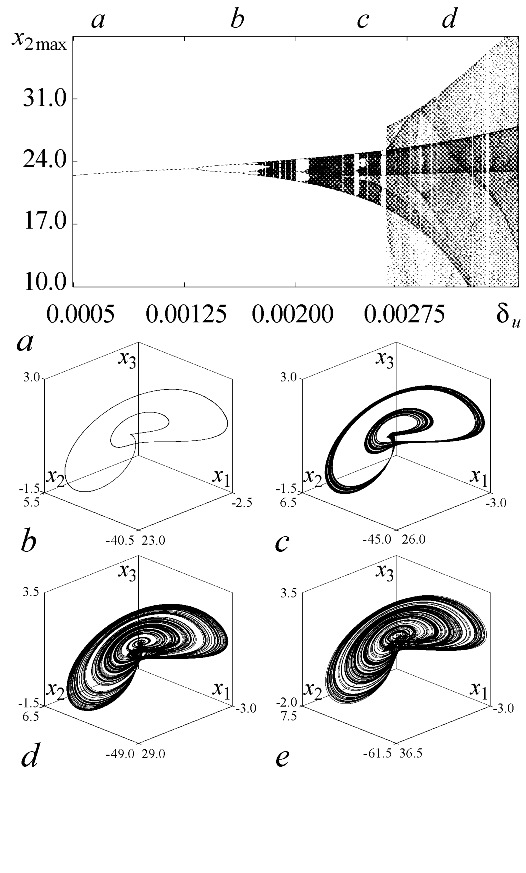
3. Mathematical Analysis of Qualitative Resonance

The previous section has given, on the basis of linear control theory and on the basis of geometrical arguments, an explanation of how can it be argued that the signals which can drive a Shil'nikov-like chaotic system to QR are those that can be obtained by perturbing the GC in the direction of its stable manifold and, on the other hand, those that drive the systems to anti-resonance can be obtained by perturbing it in the direction of its unstable manifold. Such a conjecture is confirmed by an almost exhaustive and very detailed bifurcation analysis of the phenomenon, conducted on several models combining advanced continuation techniques [5, 6] with theoretical arguments [3].

3.1. Bifurcation Analysis. The behavior of system (1) perturbed with

$$u(t) = \widehat{y}(t) + \varepsilon(t) = \widehat{y}(t) + C\big(e_p(t)\delta_p + e_s(t)\delta_s + e_u(t)\delta_u\big), \tag{4}$$

where $\widehat{y}(t)$ is the output y when the state evolves on the GC and $e_i(t)$ are the eigenvectors of the monodromy matrix tangent to the center (p), stable (s), and unstable (u) manifolds of the GC, has been analyzed with respect to the intensity (δ_i) of the perturbations in the stable and unstable direction of the GC. The generic result is reported in Fig. 3, *f* showing that indeed are the perturbations in the unstable direction which mainly lead to anti-resonance. Furthermore, a 1D bifurcation analysis of the transition from qualitative



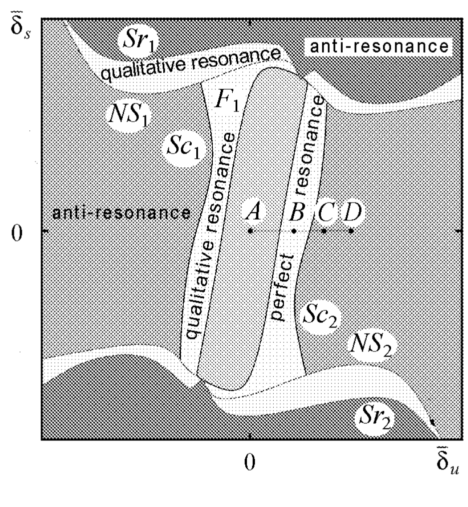


Fig. 3. (*a-e*): Route from resonance to anti-resonance: *a* – Feigenbaum's diagram of the attractors along the path to anti-resonance; *b* – perfect locking; *c* – Narrow Feigenbaum-like strange attractors;*d* – Shil'nikov-like strange attractors; *e* – swapout effect; *f* – General bifurcation diagram of the qualitative resonance with respect to perturbations in the unstable ($\bar{\delta}_u$) and stable ($\bar{\delta}_s$) directions of the generating cycle

resonance (point b in Fig. 3, *a,c*) to anti-resonance (point c in Fig. 3, *a,d*) shows that anti-resonance occurs when the perturbations in the unstable direction lead the driven state to get excessively close to the equilibrium bearing the homoclinic trajectory.

4. Remarks on Qualitative Resonance

Considering the above results together with the periodic control theory, it is possible to conclude that an externally perturbed Shil'nikov-like chaotic system acts as a state space reconstructor (periodic Kalman filter) with saturation. As long as the dynamics of the driving signal is sufficiently similar to the natural dynamics of the driven system, the latter tends to follow the driving signal attenuating its irregular fluctuations. When, on the contrary, the dynamics of the driving signal is too far from that of the driven system, the trajectory of the driven system is lead to approach the equilibrium bearing the homoclinic trajectory, reinforcing in this way its chaotic dynamics.

Finally, since the transition from QR to anti-resonance is rather sharp, the QR phenomenon can be exploited in practical problems of temporal pattern recognition [9,10].

This work was supported by the European project APEREST: IST-2001-34893 and OFES-01.0456.

Bibliography

1. *Afraimovich V.S., Verichev N.N., Rabinovich M.I.* General Synchronization // Rad. Quant. El. 1986. Vol. 29. P. 795.
2. *De FeoO.* Qualitative resonance of Shil'nikov-like strange attractors, part I: Experimental evidence // International Journal of Bifurcation & Chaos. 2004. Vol. 14. P. 873.
3. *De Feo O.* Qualitative resonance of Shil'nikov-like strange attractors, part II: Mathematical analysis // International Journal of Bifurcation & Chaos. 2004. Vol. 14. P. 893.
4. *De Feo O., Maggio G.M., Kennedy M.P.* The Colpitts oscillator: Families of periodic solutions and their bifurcations // Int. J. Bif. & Chaos. 2000. Vol. 10. P. 935.
5. *Doedel E.J., Keller H.B., Kernévez J.P.* Numerical analysis and control of bifurcation problems, part I: Bifurcation in finite dimensions // International Journal of Bifurcation and Chaos. 1991. Vol.1. P. 493.
6. *Doedel E.J., Champneys A.R., Fairgrieve T. F., Kuznetsov Y.A., Sandstede B., Wang X.J.* AUTO 97: Continuation and Bifurcation Software for Ordinary Differential Equations $withHomCont$. 1998. Computer Science Department, Concordia University, Montreal, Canada, Motreal, Quebec, Canada.
7. *Bittanti S. Bittanti S. and Colaneri P.* Periodic Control, pages 59-74, John Wiley & Sons, New York, NY, 1999.
8. *Callier F. M. and Desoer C. A.* Linear System Theory. Springer-Verlag, New York, NY, 1991.
9. *De Feo O.* Self-emergence of chaos in identifying irregular periodic behavior // Chaos. 2003. Vol. 13. P. 1205.

10. *De Feo O.* Tuning chaos synchronization and anti-synchronization for applications in temporal pattern recognition // International Journal of Bifurcation & Chaos. 2005. Accepted, to appear.

Laboratory of Nonlinear Systems,
Swiss Federal Institute of Technology Lausanne
EPFL-IC-LANOS, CH-1015 LAUSANNE, SWITZERLAND

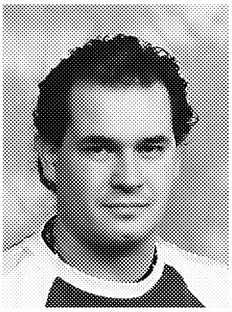
Поступила в редакцию 3.03.2005

УДК 537.86

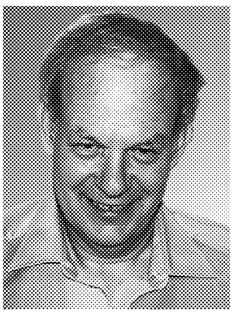
КАЧЕСТВЕННЫЙ РЕЗОНАНС ХАОТИЧЕСКИХ АТТРАКТОРОВ

Оскар де Фео и Мартин Хаслер

В работе рассматривается динамическое явление, имеющее непосредственное отношение к проблемам синхронизации и управления хаотическими динамическими системами. Рассмотрена хаотическая система вблизи гомоклинической бифуркации, управляемая внешним сигналом/системой. Показано, что если внешний сигнал качественно близок к одному из седловых циклов, «составляющих» странный аттрактор, то ведомая система синхронизуется внешним сигналом. В противном случае синхронизация не наступает. Это резонансное поведение названо качественным резонансом и исследовано теоретически и экспериментально.



Оскар де Фео – родился в 1971 году. В 1995 закончил с отличием Политехнический Институт (Милан, Италия). В 2001 получил ученую степень в Швейцарском Технологическом Институте (Лозанна, Швейцария). В настоящее время научный сотрудник и старший преподаватель кафедры нелинейных систем Швейцарского Технологического Института в Лозанне. Удостоен премии за лучшую статью на Европейской конференции по теории цепей (ECCTD'99); премии Михалевича (Mikhalevich); премии научного фонда Хорафас (Chorafas). Автор 50 научных работ, в том числе 20 статей в международных журналах. Научные интересы: моделирование и анализ связанных комплексных систем, включая ансамбли нейронов, синхронизация и бифуркации динамичесих систем, теория управления.



Мартин Хаслер – родился в 1945. В 1969 закончил Швейцарский Технологический Институт в Цюрихе, там же в 1973 получил ученую степень. В настоящее время заведует кафедрой нелинейных систем Швейцарского Технологического Института в Лозанне. Удостоен множества премий, вице-президент общества IEEE по секции электрические цепи и системы, член совета Швейцарского Национального Научного Фонда. В 1993-1995 был главным редактором журнала IEEE Transactions on Circuits and Systems, Part I. Автор более 200 научных работ, включая 4 монографии. Научные интересы включают нелинейные цепи и системы, динамику ансамблей биологических систем и синхронизацию, использование хаотической динамики для передачи информации, моделирование высокотемпературных сверхпроводников и др.