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Statistical characteristics of noise-induced intermittency in multistable systems

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The paper is devoted to the study of noise-induced intermittent behavior in multistable systems. Such task is quite important because despite of a great interest of investigators to the study of multistability and intermittency, the problem connected with the detailed understanding of the processes taking place in the multistable dynamical systems in the presence of noise and theoretical description of arising at that intermittent behavior is still remain unsolved. In present paper we analyze the noise-induced intermittency in multistable systems using the examples of model bistable system being under influence of external noise and two dissipatively coupled logistic maps subjected to additional noise. We have shown that the influence of noise on multistable system for certain values of the control parameters results in the appearance of noise-induced intermittent behavior. At that, for the found type of intermittent behavior the analytical relations for residence time distributions and dependence of the mean length of the residence times on the criticality parameter have been obtained. During the numerical simulations carried out we have found statistical characteristics for such type of intermittency for both systems, i.e. the distributions of the residence times for both coexisting stable states as well as the dependence of the mean length of the residence times for both regimes on the criticality parameter. The results of numerical simulation of intermittent behavior for systems under study have been compared with the obtained analytical regularities for noise-induced intermittency in multistable systems. At that, we have shown that numerical results and theoretical regularities are in a good agreement with each other.

Key words: intermittency, multistability, noise.

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1. Intermittency as continuous transmission from regular behaviour to chaotic is common is nature and technique [1]. It can be observed in hydrodynamics, chemical kinetics, physics of high-energy particles [2], cosmology [3] and other branches of science. There are different types of intermittent behaviour among which we distinguish three types (*intermittency types* I–III [1]), arising during the transition from periodical oscillations to chaotic ones: *on-off intermittency* [4, 5], *eyelet intermittency* [6], *ring intermittency* [7].

We must separately mark that the noise (internal or external) may also lead to intermittent behaviour [8, 9], especially in multistable systems [9, 10], when the influence of noise causes the transition from one co-existing attractor to another (this type of intermittent behaviour is also known as *noise-induced switching between attractors* [11, 12] or *noise-induced multistable intermittency* [9, 10, 13]). In spite of great interest of the scientists for the study of this type of dynamics, in our days there are many questions conduce most of all by complexity of the investigated systems, which highly obstructs (and in some cases makes impossible) the theoretical description of the process, leaving us the numerical and experimental investigation only.

2. Wide class of objects for which this type of behaviour is possible (and which in this matter remained out of sight of researches) are the systems of interacting oscillators. When the coupling value increases, such systems transit to synchronous regime, which may be characterized by multistability in the form of coexisting in-phase and non in-phase regimes [14, 15]. The influence of noise can lead to switching between these regimes and respectively to noise-induced intermittency.

For the investigation of this behaviour we study two dissipatedly coupled logistic maps [15]

$$\begin{aligned} x_{n+1} &= \lambda - x_n^2 + \varepsilon(x_n^2 - y_n^2) + D\xi_1, \\ y_{n+1} &= \lambda - y_n^2 + \varepsilon(y_n^2 - x_n^2) + D\xi_2, \end{aligned} \tag{1}$$

where ξ_1 and ξ_2 are the Gaussian stochastic processes with with zero mean and unit variance, D is the noise intensity, λ is the control parameter, ε is the coupling parameter. In [15] it is shown that when $D = 0$ (without noise) these logistic maps with certain parameters' values ($0.75 < \lambda < 1.25$) can be found in antiphase and in-phase state, which is conduced by initial conditions of the researched system. In the case of external noise influence the system demonstrates switching between in-phase and antiphase states, which can be characterized by the parameter

$$z_n = x_{2n}, \quad \text{on condition} \quad y_{2n} < y_{th}, \tag{2}$$

where $y_{th} = 0.6$ corresponds to some threshold value. In this case the parameter z_n is in fact a noise-induced element of in-phase z_k^s or antiphase z_k^a cycle of period 2 ($z_k^s < z_i^a$). Distribution of the parameter z_n has two maxima, corresponding to the elements of in-phase and antiphase cycles. The switching between the coexisting states (in-phase and antiphase) points that the system is near cusp catastrophe [16]. In this case its behaviour can be described with the help of non-dimensional potential function with two minima. After the appropriate normalization [16] this function represented as follows:

$$U(z) = \frac{z^4}{4} - \frac{z^2}{2} + bz, \tag{3}$$

where b is the asymmetry parameter. When $|b| < 2/(3\sqrt{3})$ the potential function $U(z)$ contains two local minima $z^0_{1,2}$, divided by the critical point z^*, corresponding to maximum.

Since the variable z_n, which presents the perturbed element of cycle of period 2, changes a little during one iteration, in this case, as in the case of classical intermittency of I-type [1, 17], we can pass from discrete description to continuous one

$$\dot{z} + \frac{dU\,(z)}{dz} + D\xi(t) = 0, \tag{4}$$

where $\xi(t)$ is the Gauss process with zero mean and unit variance, D is the intensity of noise. Then we can pass to stochastic differential equation (see [8, 18] for example)

$$dZ = \frac{dU\,(z)}{dz}dt + dW, \tag{5}$$

where $Z(t)$ is the random process, $W(t)$ is one-dimensional Wiener process. In its turn the equation (5) is equivalent to Fokker-Planck equation

$$\frac{\partial\rho_Z\,(z,t)}{\partial t} = \frac{\partial}{\partial z}\left[\frac{dU\,(z)}{dz}\rho_Z\,(z,t)\right] + \frac{D}{2}\frac{\partial^2\rho_Z\,(z,t)}{\partial z^2}, \tag{6}$$

where $\rho_Z\,(z,t)$ is a probability density function of a random process $Z(t)$. To obtain the statistical characteristics of system behaviour (like the distribution of residence times of coexisting regimes and dependance of mean residence times from controlling parameters) it is necessary to analyse the evolution of probability density functions $\rho_{1,2}\,(z,t)$ for two coexisting regimes, namely $\rho_1\,(z,t)$ in the area $I_1 = -\infty < z < z^*$ and $\rho_2\,(z,t)$ in the area $I_2 = z^* < z < +\infty$. Both probability densities $\rho_{1,2}\,(z,t)$ have to obey the Fokker-Planck equation (6) in their own domains $I_{1,2}$.

As in intermittent regime the phase point stays in local minimum vicinity for a long time, we can suppose that the solution for probability density functions $\rho_{1,2}(z,t)$ must be sought in a form of metastable distribution, slowly decreasing with time (see [18]), i.e.

$$\rho_{1,2}\,(z,t) = A_{1,2}\,(t)\,r\,(z)\,, \tag{7}$$

where

$$r(z) = \frac{g(z)}{\int_{-\infty}^{\infty} g(\xi)d\xi}, \quad g(\xi) = \exp\left(-\frac{2U(\xi)}{D}\right). \tag{8}$$

Here $g(\xi)$ is the stationary probability density obtained from equation (4) for the stationary case [19]; $A_{1,2}(t)$ are coefficients slowly changing with time. The explicit form of functions $A_{1,2}(t)$ may be obtained from solution of differential equation

$$\frac{dA_{1,2}}{dt} = -\frac{k}{P_{1,2}}A_{1,2}(t)r(x^*), \tag{9}$$

where k is the proportional coefficient, $P_1 = \int_{-\infty}^{x^*} r(\xi)d\xi$, $P_2 = \int_{x^*}^{+\infty} r(\xi)d\xi$ are the probabilities of appearing of the depicting point near the first and the second local minima, accordingly. It's evident that the decreasing of $A_{1,2}(t)$ with time is described by exponential law

$$A_{1,2}(t) = A_{1,2}(0)\exp\left(-\frac{kr(x^*)}{P_{1,2}}t\right), \tag{10}$$

and the exponent indexes are different for the two local minima. Distributions of residence times of coexisting regimes [18] can be found from the following expressions

$$p_1(t) = -\int\limits_{-\infty}^{x^*} \frac{\partial \rho_1(x,t)}{\partial t} dx, \quad p_2(t) = -\int\limits_{x^*}^{\infty} \frac{\partial \rho_2(x,t)}{\partial t} dx. \tag{11}$$

Taking into consideration the normalization conditions $\int_{-\infty}^{x^*} \rho_1(\xi,0)d\xi = 1$, $\int_{x^*}^{+\infty} \rho_2(\xi,0)d\xi = 1$ they are written as follows:

$$p_{1,2}(t) = \frac{1}{T_{1,2}} \exp\left(-\frac{t}{T_{1,2}}\right). \tag{12}$$

Here $T_{1,2} = \int_0^{+\infty} t p_{1,2}(t) dt$ are the mean lengths of residence times of coexisting regimes defined as

$$T_{1,2} = \frac{P_{1,2}}{kr(x^*)} = K \exp(\pm \alpha b) \left[\frac{2}{D}\left(\frac{b^4}{4} + \frac{b^2}{2}\right)\right], \tag{13}$$

K and α are some constants. We must mark that the search of distribution of regimes duration is closely related with the problem of reaching the border, which for the first time was considered in [20].

3. Within the framework of verification of the obtained theoretical estimation (5)–(13) first of all we have made the numerical study of noise-induced intermittency in the model bistable system (3)–(4) and obtained statistical characteristics for this type of behaviour. In Fig. 1, *a* one can see the distributions of the times during which the investigated system (4) stays near one stable equilibrium state. The values of control parameters are fixed. Besides, we have obtained the relation between the mean duration

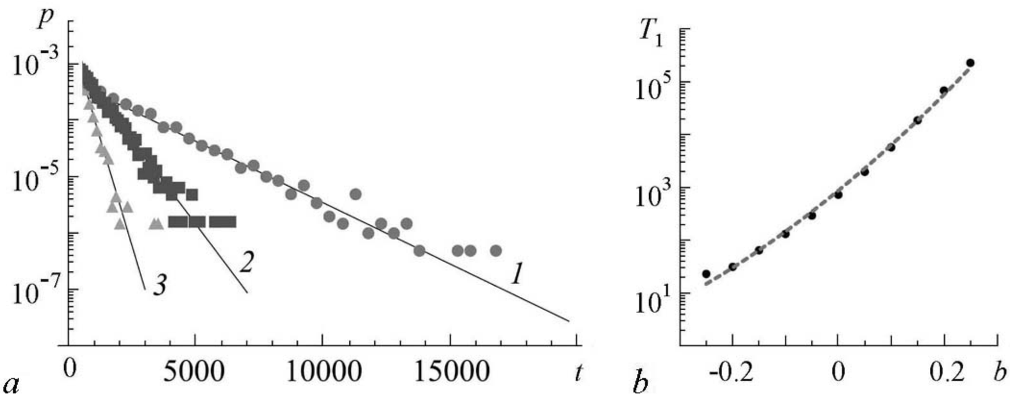

Fig. 1. *a* – distributions of the residence times for the system (4) in the vicinity of the first stable state and analytical regularity (12) corresponding to such distribution. Theoretical curves are shown by solid lines, numericaly obtained data are indicated by points. Ordibate axis is shown in logariphmic scale: *1* – $b = 0.05$, $D = 0.1$; *2* – $b = 0$, $D = 0.1$; *3* – $b = -0.05$, $D = 0.1$. *b* – dependence of the mean length of the residence times for the system (4) in the vicinity of the first stable state on the control parameter b and analytical regularity (13), corresponding to such dependence for $D = 0.1$. Theoretical curve is shown by solid line, numericaly obtained data are indicated by points. Parameters of approximation are the following: $K = 867$, $\alpha = 18.85$. Ordibate axis is shown in logariphmic scale

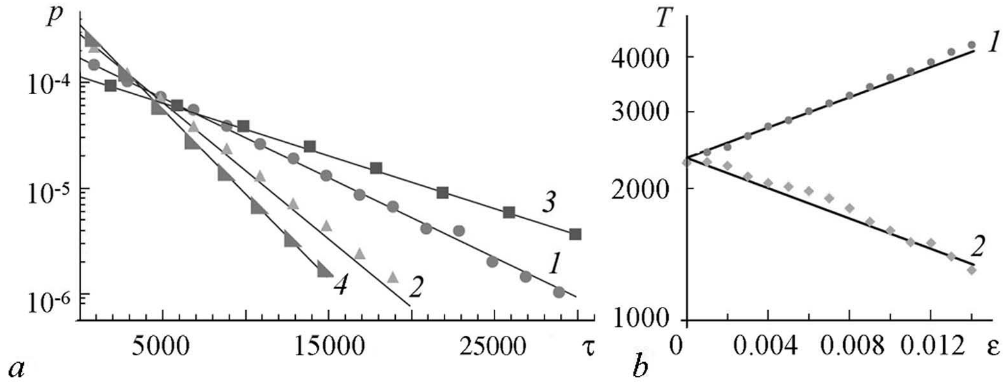

Fig. 2. *a* – distributions of the residence times for the system (1) in in-phase and antiphase states for fixed values of the control parameters $\lambda_1 = 1.05$, $\lambda_2 = 1.05$, $D = 0.06$ and analytical regularity (12), corresponding to such distributions. Theoretical curves are shown by solid lines, numerically obtained data are indicated by points. Ordibate axis is shown in logariphmic scale: *1* – in-phase case for $\varepsilon = 0.002$; *2* – antiphase case for $\varepsilon = 0.002$; *3* – in-pahse case for $\varepsilon = 0.012$; *4* – antiphase case for $\varepsilon = 0.012$. *b* – dependencies of the mean length of the residence times for the system (1) in in-pahse and antiphase states on the parameter ε and analytical regularity (13), corresponding to such dependence for $\lambda_1 = 1.05$, $\lambda_2 = 1.05$, $D = 0.06$. Theoretical curve is shown by solid line, numerically obtained data are indicated by points. Ordibate axis is shown in logariphmic scale: *1* – in-pase case; *2* – antiphase case. Parameters of approximation are the following: $K = 2350.0$, $\alpha = \pm 40.0$

of staying near the same equilibrium state and the parameter b (see Fig. 1, *b*. From Fig. 1 it is clear that the theory describing the noise-induced intermittency in multistable system, and the numerical results are in good agreement.

4. The next stage of our work is the study of intermittent behaviour of the original system of two dissipatedly coupled logistic maps (1). The corresponding distributions of times, during which the system stays in the in-phase and antiphase states with fixed parameters, are presented in Fig. 2, *a*. Also we have obtained the numerical dependance of average duration of staying is these states from the value of coupling parameter ε. This parameter allows to change the probability of staying in the in-phase and antiphase states and thus, plays the role of asymmetry parameter. The dependance is shown in Fig. 2, *b*, good agreement between the theoretical and computational results is well seen.

Conclusion. In our work we have studied the noise-induced intermittency in multistable systems using the example (4) of model bistable system under influence of external noise and the system of two dissipatedly coupled logistic maps (1). We mark that this investigation for system (1) is true for certain diapason of control parameters values (when the studied system demonstrates two coexisting regimes, synchronous and asynchronous). When the control parameters change and the period of regimes increase, the model system (4) becomes inapplicable for the analysis of this system and requires further clarification.

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